SL Paper 1

Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be defined by $f(x, y) = (x + 3y, 2x - y)$.

a. Given that *A* is the interval $\{x: 0 \leqslant x \leqslant 3\}$ and *B* is the interval $\{y: 0 \leqslant x \leqslant 4\}$ then describe *A* × *B* in geometric form. [3]

b.i. Show that the function f is a bijection. [8]

b.ii. Hence find the inverse function f^{-1}. [2]

Markscheme

a. *A* × *B* is a rectangle ***A1***

vertices at (0, 0), (3, 0), (0, 4) and (3, 4) or equivalent description ***A1***

and its interior ***A1***

Note: Accept diagrammatic answers.

[3 marks]

b.i. need to prove it is injective and surjective ***R1***

need to show if $f(x, y) = f(u, v)$ then $(x, y) = (u, v)$ ***M1***

$\Rightarrow x + 3y = u + 3v$

$2x - y = 2u - v$ ***A1***

Equation 2 – 2 Equation 1 $\Rightarrow y = v$

Equation 1 + 3 Equation 2 $\Rightarrow x = u$ ***A1***

thus $(x, y) = (u, v) \Rightarrow f$ is injective

let (s, t) be any value in the co-domain $\mathbb{R} \times \mathbb{R}$

we must find (x, y) such that $f(x, y) = (s, t)$ ***M1***

$s = x + 3y$ and $t = 2x - y$ ***M1***

$\Rightarrow y = \frac{2s - t}{7}$ ***A1***

and $x = \frac{s + 3t}{7}$ ***A1***

hence $f(x, y) = (s, t)$ and is therefore surjective

[8 marks]

b.ii. $f^{-1}(x, y) = \left(\frac{x + 3y}{7}, \frac{2x - y}{7}\right)$ ***A1A1***

[2 marks]

Examiners report

[N/A]

a.i. [N/A]
b.ii. [N/A]

The group $\{G, *\}$ is defined on the set $G = \{1, 2, 3, 4, 5, 6\}$ where $*$ denotes multiplication modulo 7.

a. Draw the Cayley table for $\{G, *\}$. [3]

b. (i) Determine the order of each element of $\{G, *\}$. [6]

(ii) Find all the proper subgroups of $\{G, *\}$.

c. Solve the equation $x * 6 * x = 3$ where $x \in G$. [3]

Markscheme

a.

*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

A3

Note: Award ***A2*** for 1 error, ***A1*** for 2 errors, ***A0*** for 3 or more errors.

[3 marks]

b. (i) We first identify 1 as the identity ***(A1)***

Order of $1 = 1$

Order of $2 = 3$

Order of $3 = 6$

Order of $4 = 3$

Order of $5 = 6$

Order of $6 = 2$ ***A3***

Note: Award ***A2*** for 1 error, ***A1*** for 2 errors, ***A0*** for more than 2 errors.

(ii) $\{1, 6\}$; $\{1, 2, 4\}$ ***A1A1***

[6 marks]

c. The equation is equivalent to

$6 * x * x = 3$ ***M1***

$x * x = 4$

$x = 2$ or 5 ***A1A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Let G denote the set of 2×2 matrices whose elements belong to $\mathbb{R}$ and whose determinant is equal to 1. Let $*$ denote matrix multiplication which may be assumed to be associative.

Let H denote the set of 2×2 matrices whose elements belong to $\mathbb{Z}$ and whose determinant is equal to 1.

a. Show that $\{G,\ *\}$ is a group. [5]

b. Determine whether or not $\{H,\ *\}$ is a subgroup of $\{G,\ *\}$. [4]

Markscheme

a. closure: let ***A***, ***B*** $\in G$

(because ***AB*** is a 2×2 matrix)

and det(***AB***) = det(***A***)det(***B***) $= 1 \times 1 = 1$ ***M1A1***

identity: the 2×2 identity matrix has determinant 1 ***R1***

inverse: let ***A*** $\in G$. Then ***A*** has an inverse because it is non-singular ***(R1)***

since ***AA***$^{-1}$ $=$ ***I***, det(***A***)det(***A***$^{-1}$) = det(***I***) = 1 therefore ***A***$^{-1} \in G$ ***R1***

associativity is assumed

the four axioms are satisfied therefore $\{G,\ *\}$ is a group ***AG***

[5 marks]

b. closure: let ***A***, ***B*** $\in H$. Then ***AB*** $\in H$ because the arithmetic involved produces elements that are integers ***R1***

inverse: ***A***$^{-1} \in H$ because the calculation of the inverse involves interchanging the elements and dividing by the determinant which is 1 ***R1***

the identity (and associativity) follow as above ***R1***

therefore $\{H,\ *\}$ is a subgroup of $\{G,\ *\}$ ***A1***

Note: Award the ***A1*** only if the first two ***R1*** marks are awarded but not necessarily the third ***R1***.

Note: Accept subgroup test.

[4 marks]

Examiners report

a. [N/A]
[N/A]

b.

The relations ρ_1 and ρ_2 are defined on the Cartesian plane as follows

$(x_1,\ y_1)\rho_1(x_2,\ y_2) \Leftrightarrow x_1^2 - x_2^2 = y_1^2 - y_2^2$

$(x_1,\ y_1)\rho_2(x_2,\ y_2) \Leftrightarrow \sqrt{x_1^2 + x_2^2} \leqslant \sqrt{y_1^2 + y_2^2}.$

a. For ρ_1 and ρ_2 determine whether or not each is reflexive, symmetric and transitive. [11]

b. For each of ρ_1 and ρ_2 which is an equivalence relation, describe the equivalence classes. [2]

Markscheme

a. ρ_1

$(x_1,\ y_1)\rho_1(x_1,\ y_1) \Rightarrow 0 = 0$ hence reflexive. ***R1***

$(x_1,\ y_1)\rho_1(x_2,\ y_2) \Rightarrow x_1^2 - x_2^2 = y_1^2 - y_2^2$

$\Rightarrow (x_1^2 - x_2^2) = -(y_1^2 - y_2^2)$

$\Rightarrow x_2^2 - x_1^2 = y_2^2 - y_1^2 \Rightarrow (x_2,\ y_2)\rho_1(x_1,\ y_1)$ hence symmetric ***M1A1***

$(x_1,\ y_1)\rho_1(x_2,\ y_2) \Rightarrow x_1^2 - x_2^2 = y_1^2 - y_2^2$ - i

$(x_2,\ y_2)\rho_1(x_3,\ y_3) \Rightarrow x_2^2 - x_3^2 = y_2^2 - y_3^2$ - ii ***M1***

$\text{i} + \text{ii} \Rightarrow x_1^2 - x_3^2 = y_1^2 - y_3^2 \Rightarrow (x_1,\ y_1)\rho_1(x_3,\ y_3)$ hence transitive ***A1***

ρ_2

$(x_1,\ y_1)\rho_2(x_1,\ y_1) \Rightarrow \sqrt{2x_1^2} \leqslant \sqrt{2y_1^2}$ This is not true in the case of (3,1)

hence not reflexive. ***R1***

$(x_1,\ y_1)\rho_2(x_2,\ y_2) \Rightarrow \sqrt{x_1^2 + x_2^2} \leqslant \sqrt{y_1^2 + y_2^2}$

$\Rightarrow \sqrt{x_2^2 + x_1^2} \leqslant \sqrt{y_2^2 – y_1^2} \Rightarrow (x_2,\ x_2)\rho_2(x_1,\ y_1)$ hence symmetric. ***A1***

it is not transitive. ***A1***

attempt to find a counterexample ***(M1)***

for example $(1,\ 0)\rho_2(0,\ 1)$ and $(0,\ 1)\rho_2(1,\ 0)$ ***A1***

however, it is not true that $(1,\ 0)\rho_2(1,\ 0)$ ***A1***

b. ρ_1 is an equivalence relation ***A1***

the equivalence classes for ρ_1 form a family of curves of the form

$y^2 - x^2 = k$ ***A1***

Examiners report

a. Most candidates attempted this question with many showing correctly that ρ_1 is an equivalence relation. Most candidates, however, were unable to find a counterexample to show that ρ_2 is not transitive although many suspected that was the case. Most candidates were unable to describe the equivalence classes.

b. Most candidates attempted this question with many showing correctly that ρ_1 is an equivalence relation. Most candidates, however, were unable to find a counterexample to show that ρ_2 is not transitive although many suspected that was the case. Most candidates were unable to describe the equivalence classes.

The permutation P is given by

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 2 & 1 \end{pmatrix}.$$

a. Determine the order of P, justifying your answer. [2]

b. Find P^2. [2]

c. The permutation group G is generated by P. Determine the element of G that is of order 2, giving your answer in cycle notation. [4]

Markscheme

a. the order is 6 ***A1***

tracking 1 through successive powers of P returns to 1 after 6 transitions (or equivalent) ***R1***

[2 marks]

b. $P^2 = (1\ 5\ 4)(2\ 6\ 3)$ or $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 1 & 4 & 3 \end{pmatrix}$ ***(M1)A1***

[2 marks]

c. since P is of order 6, P^3 will be of order 2 ***R1***

$P^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$ ***(M1)(A1)***

$P^3 = (1\ 2)(3\ 4)(5\ 6)$ ***A1***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The set P contains all prime numbers less than 2500.

The set Q is the set of all subsets of P.

The set S contains all positive integers less than 2500.

The function $f: S \to Q$ is defined by $f(s)$ as the set of primes exactly dividing s, for $s \in S$.

For example $f(4) = \{2\},\ f(45) = \{3,\ 5\}$.

a. Explain why only one of the following statements is true [4]

(i) $17 \subset P$;

(ii) $\{7,\ 17,\ 37,\ 47,\ 57\} \in Q$;

(iii) $\phi \subset Q$ and $\phi \in Q$, where ϕ is the empty set.

b. (i) State the value of $f(1)$, giving a reason for your answer. [4]

(ii) Find $n\,(f(2310))$.

c. Determine whether or not f is [4]

(i) injective;

(ii) surjective.

Markscheme

a. (i) 17 is an element not a subset of P ***R1***

(ii) 57 is not a prime number ***R1***

(iii) any demonstration that this is the true statement ***A1***

because every set contains the empty set as a subset ***R1***

[4 marks]

b. (i) $f(1) = \phi$ ***A1***

because 1 has no prime factors ***R1***

(ii) $f(2310) = f(2 \times 3 \times 5 \times 7 \times 11)\ (= \{2,\ 3,\ 5,\ 7,\ 11\})$ ***A1***

$n\,(f(2310)) = 5$ ***A1***

[4 marks]

c. (i) not injective ***A1***

because, for example, $f(2) = f(4) = \{2\}$ ***R1***

(ii) not surjective ***A1***

$f^{-1}(2,\ 3,\ 5,\ 7,\ 11,\ 13)$ does not belong to S because

$2 \times 3 \times 5 \times 7 \times 11 \times 13 > 2500$ ***R1***

Note: Accept any appropriate example.

[4 marks]

Examiners report

a. The question caused a number of problems for candidates. In part (a) a number of candidates thought part (i) was correct as they did not realise it was an element and a number thought part (ii) was correct as they did not recognise 57 as a prime number. In both of these two cases, candidates then suggested part (iii) was false giving a variety of incorrect justifications. Part (b) was more successful for most candidates with many wholly correct answers seen. Part (c) again saw many correct answers, but some candidates tried to argue the opposite, incorrect viewpoint or in other cases gave no reason for their decisions, showing a complete misunderstanding of the command term “determine”.

b. The question caused a number of problems for candidates. In part (a) a number of candidates thought part (i) was correct as they did not realise it was an element and a number thought part (ii) was correct as they did not recognise 57 as a prime number. In both of these two cases, candidates then suggested part (iii) was false giving a variety of incorrect justifications. Part (b) was more successful for most candidates with many wholly correct answers seen. Part (c) again saw many correct answers, but some candidates tried to argue the opposite, incorrect viewpoint or in other cases gave no reason for their decisions, showing a complete misunderstanding of the command term “determine”.

c. The question caused a number of problems for candidates. In part (a) a number of candidates thought part (i) was correct as they did not realise it was an element and a number thought part (ii) was correct as they did not recognise 57 as a prime number. In both of these two cases, candidates then suggested part (iii) was false giving a variety of incorrect justifications. Part (b) was more successful for most candidates with many wholly correct answers seen. Part (c) again saw many correct answers, but some candidates tried to argue the opposite, incorrect viewpoint or in other cases gave no reason for their decisions, showing a complete misunderstanding of the command term “determine”.

The group $\{G,\ *\}$ has a subgroup $\{H,\ *\}$. The relation R is defined, for $x,\ y \in G$, by xRy if and only if $x^{-1} * y \in H$.

(a) Show that R is an equivalence relation.

(b) Given that $G = \{0,\ \pm 1,\ \pm 2,\ \ldots\}$, $H = \{0,\ \pm 4,\ \pm 8,\ \ldots\}$ and $*$ denotes addition, find the equivalence class containing the number 3.

Markscheme

(a) <u>reflexive</u>

$x^{-1}x = e \in H$ ***A1***

therefore xRx and R is reflexive ***R1***

<u>symmetric</u>

Note: Accept the word commutative.

let xRy so that $x^{-1}y \in H$ ***M1***

the inverse of $x^{-1}y$ is $y^{-1}x \in H$ ***A1***

therefore yRx and R is symmetric ***R1***

<u>transitive</u>

let xRy and yRz so $x^{-1}y \in H$ and $y^{-1}z \in H$ ***M1***

therefore $x^{-1}yy^{-1}z = x^{-1}z \in H$ ***A1*** therefore xRz and R is transitive ***R1***

hence R is an equivalence relation ***AG***

[8 marks]

(b) the identity is 0 so the inverse of 3 is -3 ***(R1)***

the equivalence class of 3 contains x where $-3+x \in H$ ***(M1)***

$-3+x=4n\ (n \in \mathbb{Z})$ ***(M1)***

$x=3+4n\ (\mathrm{n} \in \mathbb{Z})$ ***A1***

Note: Accept $\{\ldots -5,\ -1,\ 3,\ 7,\ \ldots\}$ or $x \equiv 3(\bmod 4)$.

Note: If no other relevant working seen award ***A3*** for $\{3+4n\}$ or $\{\ldots -5,\ -1,\ 3,\ 7,\ \ldots\}$ seen anywhere.

[4 marks]

Examiners report

[N/A]

G is a group. The elements $a, b \in G$, satisfy $a^3=b^2=e$ and $ba=a^2b$, where e is the identity element of G.

a. Show that $(ba)^2=e$. [3]

b. Express $(bab)^{-1}$ in its simplest form. [3]

c. Given that $a \neq e$, [6]

(i) show that $b \neq e$;

(ii) show that G is not Abelian.

Markscheme

a. **EITHER**

$baba=baa^2b$ ***M1***

$=ba^3b$ ***(A1)***

$=b^2$ ***A1***

$=e$ ***AG***

OR

$baba=a^2bba$ ***M1***

$=a^2b^2a$ ***(A1)***

$=a^3$ ***A1***

$=e$ ***AG***

[3 marks]

b. $bab = a^2bb$ ***(M1)***

$= a^2$ ***(A1)***

$(bab)^{-1} = a$ ***A1***

[3 marks]

c. (i) assume $b = e$ ***M1***

then $a = a^2$ ***A1***

$\Rightarrow a = e$ which is a contradiction ***R1***

(ii) if $ab = ba$ ***M1***

then $ab = a^2b$ ***A1***

$\Rightarrow a = e$ which is a contradiction ***R1***

[6 marks]

Examiners report

a. This question was started by the majority of candidates, but only successfully completed by a few. Many candidates seemed to be aware of this style of question, but were either unable to make significant progress or manipulated the algebra in a contorted manner and hence lost valuable time. Also a number of candidates made assumptions about commutativity which were not justified. Overall, the level and succinctness of meaningful algebraic manipulation shown by candidates was disappointing.

b. This question was started by the majority of candidates, but only successfully completed by a few. Many candidates seemed to be aware of this style of question, but were either unable to make significant progress or manipulated the algebra in a contorted manner and hence lost valuable time. Also a number of candidates made assumptions about commutativity which were not justified. Overall, the level and succinctness of meaningful algebraic manipulation shown by candidates was disappointing.

c. This question was started by the majority of candidates, but only successfully completed by a few. Many candidates seemed to be aware of this style of question, but were either unable to make significant progress or manipulated the algebra in a contorted manner and hence lost valuable time. Also a number of candidates made assumptions about commutativity which were not justified. In part (c) the idea of a proof by contradiction was used by stronger candidates, but weaker candidates were often at a loss as how to start. Overall, the level and succinctness of meaningful algebraic manipulation shown by candidates was disappointing.

---

a. The set $\mathrm{S}_1 = \{2, 4, 6, 8\}$ and $\times_{10}$ denotes multiplication modulo 10. [8]

(i) Write down the Cayley table for $\{\mathrm{S}_1, \times_{10}\}$.

(ii) Show that $\{\mathrm{S}_1, \times_{10}\}$ is a group.

(iii) Show that this group is cyclic.

b. Now consider the group $\{S_1, \times_{20}\}$ where $S_2 = \{1, 9, 11, 19\}$ and $\times_{20}$ denotes multiplication modulo 20. Giving a reason, state whether or not $\{S_1, \times_{10}\}$ and $\{S_1, \times_{20}\}$ are isomorphic. [3]

Markscheme

a. (i)

$\times_{10}$	**2**	**4**	**6**	**8**
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

A2

Note: Award ***A1*** for one error.

(ii) closure: it is closed because no new elements are formed ***A1***

identity: 6 is the identity element ***A1***

inverses: 4 is self-inverse and (2, 8) form an inverse pair ***A1***

associativity: multiplication is associative ***A1***

the four group axioms are satisfied

(iii) any valid reason, *e.g.*

2 (or 8) has order 4, or 2 (or 8) is a generator ***A2***

[8 marks]

b. the groups are not isomorphic ***A1***

any valid reason, *e.g.* S_2 is not cyclic or all its elements are self-inverse ***R2***

[3 marks]

Examiners report

a. Parts (a) (i) and (a) (iii) were well answered in general. However, in (a) (ii), some candidates lost marks by not showing convincingly that $\{S_1, \times_{10}\}$ was a group. For example, in verifying the group axioms, some candidates just made bald statements such as "$\{S_1, \times_{10}\}$ is closed". This was not convincing because the question indicated that it was a group so that closure was implied by the question. It was necessary here to make some reference to the Cayley table which showed that no new elements were formed by the binary operation. To gain full marks on this style of question candidates need to clearly explain the reasoning used for deductions.

b. In (b), most candidates realised that the quickest way to establish isomorphism (or not) was to determine the order of each element. Candidates who knew that there are essentially only two different groups of order four had a slight advantage in this question.

Let $\boldsymbol{S}$ be the set of matrices given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in \mathbb{R}, ad - bc = 1$$

The relation R is defined on S as follows. Given $\boldsymbol{A}$, $\boldsymbol{B} \in S$, $\boldsymbol{ARB}$ if and only if there exists $\boldsymbol{X} \in S$ such that $\boldsymbol{A} = \boldsymbol{BX}$.

a. Show that R is an equivalence relation. [8]

b. The relationship between a , b , c and d is changed to $ad - bc = n$. State, with a reason, whether or not there are any non-zero values of n , other than 1, for which R is an equivalence relation. [2]

Markscheme

a. since $\boldsymbol{A} = \boldsymbol{AI}$ where $\boldsymbol{I}$ is the identity ***A1***

and $\det(\boldsymbol{I}) = 1$, ***A1***

R is reflexive

$\boldsymbol{ARB} \Rightarrow \boldsymbol{A} = \boldsymbol{BX}$ where $\det(\boldsymbol{X}) = 1$ ***M1***

it follows that $\boldsymbol{B} = \boldsymbol{AX}^{-1}$ ***A1***

and $\det(\boldsymbol{X}^{-1}) = \det{(\boldsymbol{X})}^{-1} = 1$ ***A1***

R is symmetric

$\boldsymbol{ARB}$ and $\boldsymbol{BRC} \Rightarrow \boldsymbol{A} = \boldsymbol{BX}$ and $\boldsymbol{B} = \boldsymbol{CY}$ where $\det(\boldsymbol{X}) = \det(\boldsymbol{Y}) = 1$ ***M1***

it follows that $\boldsymbol{A} = \boldsymbol{CYX}$ ***A1***

$\det(\boldsymbol{YX}) = \det(\boldsymbol{Y})\det(\boldsymbol{X}) = 1$ ***A1***

R is transitive

hence R is an equivalence relation ***AG***

[8 marks]

b. for reflexivity, we require $\boldsymbol{ARA}$ so that $\boldsymbol{A} = \boldsymbol{AI}$ (for all $\boldsymbol{A} \in S$) ***M1***

since $\det(\boldsymbol{I}) = 1$ and we require $\boldsymbol{I} \in S$ the only possibility is $n = 1$ ***A1***

[2 marks]

Examiners report

a. This question was not well done in general, again illustrating that questions involving both matrices and equivalence relations tend to cause problems for candidates. A common error was to assume, incorrectly, that ARB and BRC $\Rightarrow A = BX$ and $B = CX$, not realizing that a different "x" is required each time. In proving that R is an equivalence relation, consideration of the determinant is necessary in this question although many candidates neglected to do this.

b. In proving that R is an equivalence relation, consideration of the determinant is necessary in this question although many candidates neglected to do this.

Consider the set $S = \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$ under the operation of addition modulo 6, denoted by $+_6$.

a. Construct the Cayley table for $\{S,\ +_6\}$. [2]

b. Show that $\{S,\ +_6\}$ forms an Abelian group. [5]

c. State the order of each element. [2]

d. Explain whether or not the group is cyclic. [2]

Markscheme

a.

	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

A2

Note: ***A1*** for one or two errors in the table, ***A0*** otherwise.

b. closed no new elements ***A1***

0 is identity (since $0 + a = a + 0 = a,\ a \in S$) ***A1***

0, 3 self inverse, $1 \Leftrightarrow 5$ inverse pair, $2 \Leftrightarrow 4$ inverse pair ***A1***

all elements have an inverse

associativity is assumed over addition ***A1***

since symmetry on leading diagonal in table or commutativity of addition ***A1***

$\Rightarrow \{S,\ +_6\}$ is an Abelian group ***AG***

c.

Element	Order
0	1
1	6
2	3
3	2
4	3
5	6

A2

Note: ***A1*** for one or two errors in the table, ***A0*** otherwise.

d. since there is an element with order 6 **OR** 1 or 5 are generators ***R1***

the group is cyclic ***A1***

Examiners report

a. This question was well answered in general although some candidates showed only commutativity, not realising that they also had to prove that it was a group.

b. This question was well answered in general although some candidates showed only commutativity, not realising that they also had to prove that it was a group.

c. This question was well answered in general although some candidates showed only commutativity, not realising that they also had to prove that it was a group.

d. This question was well answered in general although some candidates showed only commutativity, not realising that they also had to prove that it was a group.

Prove that the function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ defined by $f(x,\ y)=(2x+y,\ x+y)$ is a bijection.

Markscheme

to be a bijection it must be injective and surjective ***R1***

Note: This ***R1*** may be awarded at any stage

suppose $f(x,\ y)=f(u,\ v)$ ***M1***

$2x+y=2u+v \quad (\text{ - i})$

$x+y=u+v \quad (\text{ - ii})$

$\text{i - ii} \Rightarrow x=u$

$\text{i - 2(ii)} \Rightarrow -y=-v$

$\Rightarrow x=u,\ y=v$ ***A1***

thus $(x,\ y)=(u,\ v)$ hence injective ***A1***

let $2x+y=s \quad (\text{ - i})$

$x+y=t \quad (\text{ - ii})$ ***M1***

$\text{i - ii } x=s-t$

$\Rightarrow y=2t-s$

both x and y are integer if s and t are integer ***R1***

hence it is surjective ***A1***

hence f is a bijection ***AG***

Note: Accept a valid argument based on matrices

Examiners report

Most candidates were able to show that f was an injection although some candidates appear to believe that it is sufficient to show that $f(x,\ y)$ is unique. A significant minority failed to show that f is a surjection and most candidates failed to note that it had to be checked that all values were integers. Some candidates introduced a matrix to define the transformation which was often a successful alternative method.

---

a. Show that the set S of numbers of the form $2^m\times 3^n$, where $m,n\in\mathbb{Z}$, forms a group $\{S,\times\}$ under multiplication. [6]

b. Show that $\{S, \times\}$ is isomorphic to the group of complex numbers $m + n\mathrm{i}$ under addition, where $m, n \in \mathbb{Z}$. [6]

Markscheme

a. Closure: Consider the numbers $2^{m_1} \times 3^{m_1}$ and $2^{m_2} \times 3^{n_2}$ where ***M1***

$m_1, m_2, n_1, n_2, \in \mathbb{Z}$. Then,

Product $= 2^{m_1+m_2} \times 3^{n_1+n_2}$ which $\in S$ ***A1***

Identity: $2^0 \times 3^0 = 1 \in S$ ***A1***

Since $(2^m \times 3^n) \times (2^{-m} \times 3^{-n}) = 1$ and $2^{-m} \times 3^{-n} \in S$ ***R1***

then $2^{-m} \times 3^{-n}$ is the inverse. ***A1***

Associativity: This follows from the associativity of multiplication. ***R1***

[6 marks]

b. Consider the bijection

$f(2^m \times 3^n) = m + n\mathrm{i}$ ***(M1)***

Then

$f(2^{m_1} \times 3^{n_1}) \times (2^{m_2} \times 3^{n_2}) = f(2^{m_1+m_2} \times 3^{n_1+n_2})$ ***M1A1***

$= m_1 + m_2 + (n_1 + n_2)\mathrm{i}$ ***A1***

$= (m_1 + n_1\mathrm{i}) + (m_2 + n_2\mathrm{i})$ ***(A1)***

$= f(2^{m_1} \times 3^{n_1}) + f(2^{m_2} \times 3^{n_2})$ ***A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]

The relation R is defined on the set $\mathbb{Z}$ by aRb if and only if $4a + b = 5n$, where $a, b, n \in \mathbb{Z}$.

a. Show that R is an equivalence relation. [8]

b. State the equivalence classes of R . [3]

Markscheme

a. $4a + b = 5n$ for $a, b, n \in \mathbb{Z}$

reflexive:

$4a + a = 5a$ so aRa , and R is reflexive ***A1***

symmetric:

$4a + b = 5n$

$4b + a = 5b - b + 5a - 4a$ ***M1***

$= 5b + 5a - (4a + b)$ ***A1***

$= 5m$ so bRa , and R is symmetric ***A1***

transitive:

$4a + b = 5n$ ***M1***

$4b + c = 5k$ ***M1***

$4a + 5b + c = 5n + 5k$ ***A1***

$4a + c = 5(n + k - b)$ so aRc , and R is transitive ***A1***

therefore R is an equivalence relation ***AG***

[8 marks]

b. equivalence classes are

$\{\ldots, -10, -5, 0, 5, 10, \ldots\}$ ***(M1)***

$\{\ldots, -9, -4, 1, 6, 11, \ldots\}$

$\{\ldots, -8, -3, 2, 7, 12, \ldots\}$

$\{\ldots, -7, -2, 3, 8, 13, \ldots\}$

$\{\ldots, -6, -1, 4, 9, 14, \ldots\}$

or $\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 4\rangle\}$ ***A2***

Note: Award ***A2*** for all classes, ***A1*** for at least 2 correct classes.

[3 marks]

Examiners report

a. Part (a) was generally well done but not always in the most direct manner.

b. Too many missed the equivalence classes in part (b).

The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ is defined by $f(x,\ y) = \left(xy,\ \frac{x}{y}\right)$.

Prove that f is a bijection.

Markscheme

we need to show that f is injective and surjective ***(R1)***

Note: Award ***R1*** if seen anywhere in the solution.

<u>injective</u>

let $(a,\ b)$ and $(c,\ d) \in \mathbb{R}^+ \times \mathbb{R}^+$, and let $f(a,\ b) = f(c,\ d)$ ***M1***

it follows that

$ab = cd$ and $\frac{a}{b} = \frac{c}{d}$ ***A1***

multiplying these equations,

$a^2 = c^2 \Rightarrow a = c$ and therefore $b = d$ **A1**
since $f(a,\ b) = f(c,\ d) \Rightarrow (a,\ b) = (c,\ d),\ f$ is injective ***R1***

Note: Award ***R1*** if stated anywhere as needing to be shown.

surjective
let $(p,\ q) \in \mathbb{R}^+ \times \mathbb{R}^+$
consider $f(x,\ y) = (p,\ q)$ so $xy = p$ and $\frac{x}{y} = q$ ***M1A1***
multiplying these equations,
$x^2 = pq$ so $x = \sqrt{pq}$ and therefore $y = \sqrt{\frac{p}{q}}$ ***A1***
so given $(p,\ q) \in \mathbb{R}^+ \times \mathbb{R}^+,\ \exists (x,\ y) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $f(x,\ y) = (p,\ q)$ which shows that f is surjective ***R1***

Note: Award ***R1*** if stated anywhere as needing to be shown.

f is therefore a bijection
[9 marks]

Examiners report

[N/A]

The transformations T_1, T_2, T_3, T_4, in the plane are defined as follows:

T_1 : A rotation of 360° about the origin
T_2 : An anticlockwise rotation of 270° about the origin
T_3 : A rotation of 180° about the origin
T_4 : An anticlockwise rotation of 90° about the origin.

The transformation T_5 is defined as a reflection in the x-axis.

The transformation T is defined as the composition of T_3 followed by T_5 followed by T_4.

a. Copy and complete the following Cayley table for the transformations of T_1, T_2, T_3, T_4, under the operation of composition of transformations. [2]

	T_1	T_2	T_3	T_4
T_1	T_1	T_2	T_3	T_4
T_2	T_2			
T_3	T_3			
T_4	T_4			

b.i. *Show that* T_1, T_2, T_3, T_4 under the operation of composition of transformations form a group. Associativity may be assumed. [3]

b.ii. Show that this group is cyclic. [1]

c. Write down the 2 × 2 matrices representing T_3, T_4 and T_5. [3]

d.i. Find the 2 × 2 matrix representing T. [2]

d.ii. Give a geometric description of the transformation T. [1]

Markscheme

a.

	T_1	T_2	T_3	T_4
T_1	T_1	T_2	T_3	T_4
T_2	T_2	T_3	T_4	T_1
T_3	T_3	T_4	T_1	T_2
T_4	T_4	T_1	T_2	T_3

A2

[2 marks]

Note: Award ***A1*** for 6, 7 or 8 correct.

b.i. the table is closed – no new elements ***A1***

T_1 is the identity ***A1***

T_3 (and T_1) are self-inverse; T_2 and T_4 are an inverse pair. Hence every element has an inverse ***A1***

hence it is a group ***AG***

[3 marks]

b.ii. all elements in the group can be generated by T_2 (or T_4) ***R1***

hence the group is cyclic ***AG***

[1 mark]

c.

T_3 is represented by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ***A1***

T_4 is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ***A1***

T_5 is represented by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ***A1***

[3 marks]

d.i. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ ***(M1)A1***

Note: Award ***M1A0*** for multiplying the matrices in the wrong order.

[2 marks]

d.ii. a reflection in the line $y = -x$ ***A1***

[1 mark]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
c. [N/A]
d.i. [N/A]
d.ii. [N/A]

$\{G,\ *\}$ is a group of order N and $\{H,\ *\}$ is a proper subgroup of $\{G,\ *\}$ of order n.

(a) Define the right coset of $\{H,\ *\}$ containing the element $a \in G$.

(b) Show that each right coset of $\{H,\ *\}$ contains n elements.

(c) Show that the union of the right cosets of $\{H,\ *\}$ is equal to G.

(d) Show that any two right cosets of $\{H,\ *\}$ are either equal or disjoint.

(e) Give a reason why the above results can be used to prove that N is a multiple of n.

Markscheme

(a) the right coset containing a has the form $\{ha|h \in H\}$ ***A1***

[1 mark]

Note: From here on condone the use of left cosets.

(b) let b, c be distinct elements of H. Then, given $a \in G$, by the Latin square property of the Cayley table, ba and ca are distinct ***A1***
therefore each element of H corresponds to a unique element in the coset which must therefore contain n elements ***R1***
[2 marks]

(c) let d be any element of G. Then since H contains the identity e, $ed = d$ will be in a coset ***R1***
therefore every element of G will be contained in a coset which proves that the union of all the cosets is G ***R1***
[2 marks]

(d) let the cosets of b and c $(b,\ c \in G)$ contain a common element so that
$pb = qc$ where $p,\ q \in H$. Let r denote any other element $\in H$ ***M1***
then
$rb = rp^{-1}qc$ ***A1***
since $rp^{-1}q \in H$, this shows that all the other elements are common and the cosets are equal ***R1***
since not all cosets can be equal, there must be other cosets which are disjoint ***R1***
[4 marks]

(e) the above results show that G is partitioned into a number of disjoint subsets containing n elements so that N must be a multiple of n ***R1***
[1 mark]

Examiners report

[N/A]

The set S contains the eight matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where a, b, c can each take one of the values $+1$ or -1 .

a. Show that any matrix of this form is its own inverse. [3]

b. Show that S forms an Abelian group under matrix multiplication. [9]

c. Giving a reason, state whether or not this group is cyclic. [1]

Markscheme

a. $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$ ***A1M1***

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ***A1***

this shows that each matrix is self-inverse

[3 marks]

b. closure:

$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & 0 & 0 \\ 0 & b_1b_2 & 0 \\ 0 & 0 & c_1c_2 \end{pmatrix}$ ***M1A1***

$= \begin{pmatrix} a_3 & 0 & 0 \\ 0 & b_3 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$

where each of a_3, b_3, c_3 can only be ± 1 ***A1***

this proves closure

identity: the identity matrix is the group identity ***A1***

inverse: as shown above, every element is self-inverse ***A1***

associativity: this follows because matrix multiplication is associative ***A1***

S is therefore a group ***AG***

Abelian:

$\begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix} = \begin{pmatrix} a_2a_1 & 0 & 0 \\ 0 & b_2b_1 & 0 \\ 0 & 0 & c_2c_1 \end{pmatrix}$ ***A1***

$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & 0 & 0 \\ 0 & b_1b_2 & 0 \\ 0 & 0 & c_1c_2 \end{pmatrix}$ ***A1***

Note: Second line may have been shown whilst proving closure, however a reference to it must be made here.

we see that the same result is obtained either way which proves commutativity so that the group is Abelian ***R1***

[9 marks]

c. since all elements (except the identity) are of order 2, the group is not cyclic (since S contains 8 elements) ***R1***

[1 mark]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

---

a. Prove that the number 14641 is the fourth power of an integer in any base greater than 6. [3]

b. For $a, b \in \mathbb{Z}$ the relation aRb is defined if and only if $\frac{a}{b} = 2^k$, $k \in \mathbb{Z}$. [8]

(i) Prove that R is an equivalence relation.

(ii) List the equivalence classes of R on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Markscheme

a. 14641 (base $a > 6$) $= a^4 + 4a^3 + 6a^2 + 4a + 1$, ***M1A1***

$= (a + 1)^4$ ***A1***

this is the fourth power of an integer ***AG***

[3 marks]

b. (i) aRa since $\frac{a}{a} = 1 = 2^0$, hence R is reflexive ***A1***

$aRb \Rightarrow \frac{a}{b} = 2^k \Rightarrow \frac{b}{a} = 2^{-k} \Rightarrow bRa$

so R is symmetric ***A1***

aRb and $bRc \Rightarrow \frac{a}{b} = 2^m, m \in \mathbb{Z}$ and $bRc \Rightarrow \frac{b}{c} = 2^n$, $n \in \mathbb{Z}$ ***M1***

$\Rightarrow \frac{a}{b} \times \frac{b}{c} = \frac{a}{c} = 2^{m+n}$, $m + n \in \mathbb{Z}$ ***A1***

$\Rightarrow aRc$ so transitive ***R1***

hence R is an equivalence relation ***AG***

(ii) equivalence classes are $\{1, 2, 4, 8\}$, $\{3, 6\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$ ***A3***

Note: Award ***A2*** if one class missing, ***A1*** if two classes missing, ***A0*** if three or more classes missing.

[8 marks]

Examiners report

a. This was not difficult but a surprising number of candidates were unable to do it. Care with notation and logic were lacking.

b. The question was at first straightforward but some candidates mixed up the properties of an equivalence relation with those of a group. The idea of an equivalence class is still not clearly understood by many candidates so that some were missing.

The group $\{G,+\}$ is defined by the operation of addition on the set $G=\{2n|n\in\mathbb{Z}\}$.

The group $\{H,+\}$ is defined by the operation of addition on the set $H=\{4n|n\in\mathbb{Z}\}$

Prove that $\{G,+\}$ and $\{H,+\}$ are isomorphic.

Markscheme

consider the function $f:G\to H$ defined by $f(g)=2g$ where $g\in G$ ***A1***

given $g_1, g_2\in G, f(g_1)=f(g_2)\Rightarrow 2g_1=2g_2\Rightarrow g_1=g_2$ (injective) ***M1***

given $h\in H$ then $h=4n$, so $f(2n)=h$ and $2n\in G$ (surjective) ***M1***

hence ***f*** is a bijection ***A1***

then, for $g_1, g_2\in G$

$f(g_1+g_2)=2(g_1+g_2)$ ***A1***

$f(g_1)+f(g_2)=2g_1+2g_2$ ***A1***

it follows that $f(g_1+g_2)=f(g_1)+f(g_2)$ ***R1***

which completes the proof that $\{G,+\}$ and $\{H,+\}$ are isomorphic ***AG***

[7 marks]

Examiners report

[N/A]

---

a. Use the Euclidean algorithm to find $\gcd(162,\ 5982)$. [4]

b. The relation R is defined on $\mathbb{Z}^+$ by nRm if and only if $\gcd(n,\ m)=2$. [7]

(i) By finding counterexamples show that R is neither reflexive nor transitive.

(ii) Write down the set of solutions of $nR6$.

Markscheme

a. $5982=162\times 36+150$ ***M1A1***

$162=150\times 1+12$ ***A1***

$150=12\times 12+6$

$12=6\times 2+0\Rightarrow \gcd$ is 6 ***A1***

[4 marks]

b. (i) for example, $\gcd(4,\ 4)=4$ ***A1***

$4\neq 2$ ***R1***

so R is not reflexive ***AG***

for example

$\gcd(4,\ 2) = 2$ and $\gcd(2,\ 8) = 2$ ***M1A1***

but $\gcd(4,\ 8) = 4\ (\ne 2)$ ***R1***

so R is not transitive ***AG***

(ii) **EITHER**

even numbers ***A1***

not divisible by 6 ***A1***

OR

$\{2 + 6n : n \in \mathbb{N}\} \cup \{4 + 6n : n \in \mathbb{N}\}$ ***A1A1***

OR

$2,\ 4,\ 8,\ 10,\ \ldots$ ***A2***

[7 marks]

Examiners report

a. This was a successful question for many students with many wholly correct answers seen. Part (a) was successfully answered by most candidates and those candidates usually had a reasonable understanding of how to complete part (b). A number were not fully successful in knowing how to explain their results.

b. This was a successful question for many students with many wholly correct answers seen. Part (a) was successfully answered by most candidates and those candidates usually had a reasonable understanding of how to complete part (b). A number were not fully successful in knowing how to explain their results.

A sample of size 100 is taken from a normal population with unknown mean μ and known variance 36.

Another investigator decides to use the same data to test the hypotheses H_0: $\mu = 65$, H_1: $\mu = 67.9$.

a. An investigator wishes to test the hypotheses H_0: $\mu = 65$, H_1: $\mu > 65$. [3]

He decides on the following acceptance criteria:

Accept H_0 if the sample mean $\bar{x} \le 66.5$

Accept H_1 if $\bar{x} > 66.5$

Find the probability of a Type I error.

b.i. She decides to use the same acceptance criteria as the previous investigator. Find the probability of a Type II error. [3]

b.ii. Find the critical value for $\bar{x}$ if she wants the probabilities of a Type I error and a Type II error to be equal. [3]

Markscheme

a. $\bar{X} \sim \mathrm{N}\left(\mu,\ \frac{\sigma^2}{n}\right)$

$\bar{X} \sim \mathrm{N}\left(65,\ \frac{36}{100}\right)$ ***(A1)***

P(Type I Error) $= \mathrm{P}\left(\bar{X} > 66.5\right)$ ***(M1)***

= 0.00621 ***A1***

[3 marks]

b.i. P(Type II Error) = P(accept $H_0|H_1$ is true)

$= \mathrm{P}\left(\bar{X} \leqslant 66.5\,|\mu = 67.9\right)$ ***(M1)***

$= \mathrm{P}\left(\bar{X} \leqslant 66.5\right)$ when $\bar{X} \sim \mathrm{N}\left(67.9,\ \frac{36}{100}\right)$ ***(M1)***

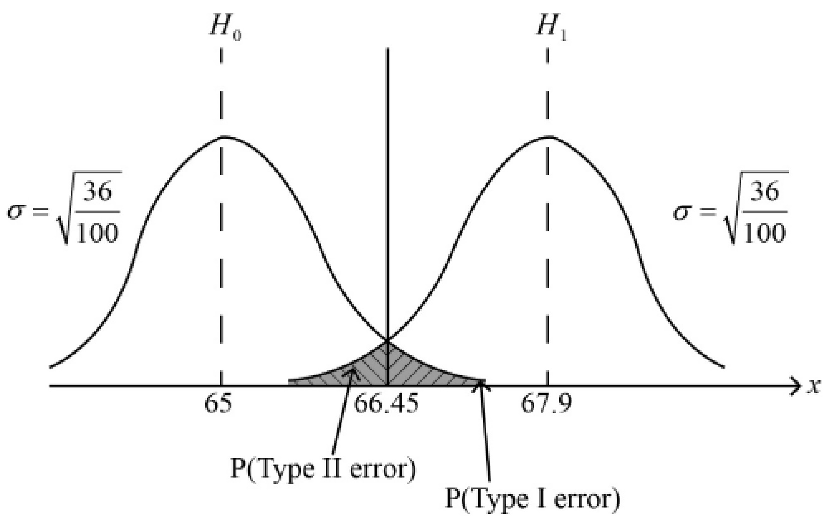
= 0.00982 ***A1***

[3 marks]

b.ii. the variances of the distributions given by H_0 and H_1 are equal, **(R1)**

by symmetry the value of $\bar{x}$ lies midway between 65 and 67.9 ***(M1)***

$\Rightarrow \bar{x} = \frac{1}{2}(65 + 67.9) = 66.45$ ***A1***



[3 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]